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Transition from laminar to turbulent flow

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A NATO Advanced Study Institute on the topic of transition from laminar flow to turbulence was held at Imperial College, London, from 1 to 6 July 1968. Each morning's session was started with a one-hour general lecture, and was followed by five or six half-hour lectures interspaced with discussion periods. The main lecturers were C. C. Lin (general survey), S. Rosenblat (stability of time-dependent flows), L. S. G. Kovasznay (turbulent, non-turbulent interfaces), L. E. Scriven (free surface effects) and A. A. Townsend (shear turbulence). The idea of the meeting was to bring forth and to discuss current ideas in the subject, both from the point of view of developments *out of* laminar flow and from that of developments *into* real turbulence. To this end speakers were chosen to introduce a variety of topics ranging from laminar-flow instabilities (with emphasis on aspects at present imperfectly understood), through non-linear effects to the processes affecting turbulence itself.

Many ideas recurred throughout the meeting, both at lectures and in discussion periods. This is true, for example, of several relevant points forcefully made by C. C. Lin. For this reason the present account does not attempt to describe the proceedings of the meeting in chronological order, but rather takes an overall view of the subject matter and points to the areas of agreement and of controversy in relation to various problems.

1. General survey of the problem

The scene was set in a truly admirable lecture by Professor C. C. Lin of the Massachusetts Institute of Technology. Right at the outset he revealed his current interest in instabilities which yield not transition to turbulence, but the spiral arms of the galaxies! He was able, however, to use beautiful photographs of galaxies to illustrate some of the points which he made about our less grand, but equally exciting, problem.

With one eye (possibly) on some form of turbulence, it is usual to discuss the stability of and to experiment upon, laminar flows which are relevant to turbulent motion under similar geometrical and dynamical conditions. Theoretical discussions are often concerned with the normal modes, or characteristic oscillations of laminar flows, and it is known that such normal modes *can* be realized in experiment; examples which spring to mind are G. I. Taylor's (1923) observations on rotating-cylinder flows and those of G. B. Schubauer & H. K. Skramstad (1943) on boundary layers. However, the reproduction in experiment of the relatively simple normal modes is one thing, and turbulence is another! Are there situations where the normal modes of laminar flow, or of some quasi-turbulent flow, play an important structural part in fully developed turbulence itself? Clearly the existence of such relationships would give added weight to normal-mode studies of the instability of laminar flow. Much current work on turbulence in shear flows is based on this idea, for example, that of Landahl (1967). Lin stressed the importance of large-scale structures persisting from laminar flow, and argued that this gave added weight to studies of basic non-linear (but not yet random) processes.

Lin also drew attention to the idea of what he called 'fast' and 'slow' transitions, a classification bearing a strong similarity to that of Coles (1965), who described 'catastrophic transition' and 'transition by spectral evolution'. A typical 'fast' transition would be that arising from Tollmien–Schlichting waves in a boundary layer where, within the spatial confines of a single experiment, a flow can be seen to evolve quickly from laminar flow to turbulence. In a wake, however, a 'slow' transition occurs and regular finite-amplitude motions in near equilibrium (vortex streets) play a dominant role; another example, of course, is the occurrence of Taylor vortices between two concentric rotating circular cylinders when the inner is rotating and the outer is at rest.

Many years ago, Landau (1944) brought forward the idea of transition occurring through successive instabilities. In using simple modes to describe turbulence Landau mentioned the importance of the 'phase'; Lin suggested that, if the phases are randomly distributed, comparatively few modes could be used to describe turbulence. He asked whether 'bursts' or 'spots' of turbulence, which play an important role in the late stages of transition in boundary layers, are responsible for momentum transfer in transitional and turbulent boundary layers, or whether they simply dissipate energy; and what is the relative importance of (i) 'bursts' of high-frequency oscillations, and (ii) developing non-linear modes? One is tempted to relate (i) and (ii) to 'fast' and 'slow' respectively.

In discussing the effects of non-linearity it is natural also to inquire about the relative importance of non-linear and viscous forces. Benney & Bergeron (1968) have developed an analysis which shows that, for a single mode, non-linear terms dominate over viscosity in the neighbourhood of the 'critical layer', provided that $\epsilon R^{\frac{2}{3}}$ is large compared with 1, where ϵ is a non-dimensional velocity amplitude and R is a Reynolds number of the boundary layer. However viscous shear layers appear at the boundaries of the so-called Kelvin's 'cat's eyes'. Lin explained how Benney & Bergeron used this theory to account for certain ambiguities in the properties of finite amplitude oscillations in mixing layers, as calculated by Schade (1964) and Stuart (1967). The basic idea as explained by Lin, and later explained at greater length by Bergeron, is that in the neighbourhood of the critical layer we can form a local Reynolds number based on the non-dimensional velocity ϵ and on the characteristic thickness $R^{-\frac{1}{3}}$; this yields $\epsilon R^{\frac{2}{3}}$, as quoted above.

In the discussion which followed it was emphasized by T. Brooke Benjamin that jets have a more organized turbulence than do flows in a pipe, indicating the importance of non-linear modes in the former case. It was also noted that in mixing layers viscosity has an important role to play in growing oscillations but that this viscous effect disappears in equilibrium (Stuart 1967), leaving non-linear effects in dominance.

2. Developments of instabilities at small amplitude

One or two newer developments were described and will be outlined here. S. Rosenblat gave a lecture on the stability of basic flows which themselves depend upon time. An interesting example (though not described at the meeting) occurs when a can, containing water having a free surface, is oscillated vertically. Unstable oscillations of the free surface may be stimulated by appropriate vertical oscillations of the can, and the governing equation is that of Mathieu. A more complex example, not governed by such simple mathematics, occurs when the inner of two long concentric circular cylinders has an angular speed $(\Omega_1+\omega_1\cos nt)$; by appropriate choice of ω_1 and n the flow can be stabilized against the occurrence of Taylor vortices (as Donnelly (1964) has shown experimentally).

There are difficulties in quantifying the statement of stability or instability, but one satisfactory method is to consider the ratio of the perturbation magnitude to the magnitude of the given flow. If the latter is $U(y_1t) = \overline{u}(y)A(t)$ in the x direction, the theory of Shen (1961) shows that x periodic inviscid disturbances are stabilized by a monotonically increasing $A(t)$, but destabilized by a monotonically decreasing $A(t)$. The governing equation for a disturbance $\phi e^{i\alpha x}$ is $(\partial/\partial t+i\alpha U)(\phi_{yy}-\alpha^2\phi)+i\alpha U_{yy}\phi = 0$. In inviscid flow, the disturbance is proportional to

$$B(t) = \exp\left[-i\alpha c\int^t A(s)\,ds\right], \quad \text{if} \quad U = \overline{u}(y)\,A(t),$$

where c is a complex number and α is the wave-number in the x direction. The result quoted above follows when we consider $(d/dt)(B/A)$. If $A(t)$ is *periodic* in time B is bounded, so that no true instability results. However, as pointed out in the discussion, the amplitude may become very large, and quasi-steady instability theoreies for *some* time interval may be relevant in giving, if not true instability, at least growth to large-amplitude. By then non-linearity would be important.

The class of flows described in the previous paragraph is rather special. A more typical example is the Stokes shear layer

$$U(y,t) = U_0 e^{-y\sqrt{(\omega/2\nu)}}\cos(\omega t - y\sqrt{(\omega/2\nu)}),$$

which has a phase dependent on y. Such a flow may be represented by the sum of two terms like $\overline{u}(y)\,A(t)$, and S. Rosenblat stated that such flows are destabilized by the phase dependence on y (if vorticity is present). It is necessary to study a system of equations of Mathieu type. Problems when a mean flow is also present have been studied by Kelly (1965), who showed that parametric resonances can occur.

Dr M. Gaster discussed the development of localized regions of fluctuation by synthesis from Fourier modes of the Orr–Sommerfeld stability problem for steady flows; he was able to show how wave packets, according to this linearized theory, evolve within the confines of a wedge stemming from the point of origin. For a Blasius layer the wedge semi-angle is about 11° which is fairly close to the angles of wedges enveloping 'spots' of turbulence, but observed velocities of such spots are higher than is predicted by linearized theory. It would be of great interest to inject non-linearities into the mathematical descriptions of local wave packets, but this has not so far been done.

In his lecture on the development of unstable waves of the form $e^{i\alpha(x-ct)}$ in free shear layers, Dr A. Michalke emphasized the importance of modes with complex α as being especially relevant in comparison with experiment. Experimentally, he could follow such growing modes in mixing regions over a distance of about four wavelengths before they broke up into turbulence. Dr Michalke also spoke of the role of vorticity conservation in two-dimensional flow, and commented upon the violation of this brought about by linearization.

Professor A. Faller introduced the topic of the instabilities of Ekman layers. Such flows have long been known to exhibit instabilities (type I) associated with inflexion points (vorticity maxima) of the velocity profile in the direction of propagation of wavy disturbances. Another mode of instability (type II), driven by Coriolis forces associated with the perturbation, is now known to produce instability and generate waves at a lower Reynolds number than the vorticity mechanism. Whereas the type I modes are frequently stationary, those of type II are travelling waves. Further experimental work has shown the type II modes to occur on a rotating disk (Gregory, Stuart & Walker 1955), and Professor Faller has identified a secondary instability of the type II waves, possibly due to centrifugal effects. Applications to the thermal wind indicate that vortices could be produced by the type II mechanism, in accordance with observation.

Dr S. H. Davis described a rigorous perturbation scheme for deriving a 'principle of exchange of stabilities' for a wide class of instability problems, and has treated several hitherto unsolved cases associated with thermal convection and rotating cylinder instabilities. In brief this principle establishes that instability, when it occurs, does so as a monotonic growth and not as oscillations.

3. Non-linear processes affecting normal modes

Dr H. Sato gave a very comprehensive and interesting account of his experiments on transition processes in wakes, jets and mixing regions. For several reasons he regarded the wake as the flow, typical of those with no solid boundaries, where transition can be studied in the most controlled manner. He first described experiments in which the wake was stimulated by sound of a given single frequency, and noted that in the downstream region the amplitude of resulting wake fluctuations was independent of the forcing amplitude, suggesting that a 'natural' amplitude of the non-linear wake had been attained; frequency doubling took place due to harmonic generation. There was a dip in longitudinal velocity on the centre line.

In other, more detailed experiments, Dr Sato described how the wake responded when two frequencies, one 10% greater than the other, were superimposed. These two frequencies (n_1 and n_2) were chosen to give a maximum interaction effect between the modes. There was a maximum r.m.s. fluctuation velocity on the centre line of the wake, which was found to be principally associated with the frequency $n_1 - n_2$, in accordance with the idea that the fundamental linearized modes are antisymmetrical, whereas the harmonics (and mixed harmonics) of n_1 and n_2 are symmetrical. Harmonics of $n_1 - n_2$ also became important elsewhere in the flow. The use of these two frequencies, suitably chosen, gave a much closer representation of natural transition than did the use of one mode, presumably due to the increased spread of energy in the frequency spectrum. Notable was the occurrence of a saw-toothed wave, with superimposed random oscillations, before transition.

Discussion centred on the mechanism by which shear oscillations in a wake are generated by sound. The common view that the shear oscillation takes the same frequency as the sound, but picks out its natural shear wavelength was opposed by the idea that it is the diffraction of the sound wave at the trailing edge of the airfoil (producing the wake flow), which yields the appropriate smaller length scale of the shear oscillations. More work clearly is needed on such processes.

Professor I. Tani, in a paper on transition in boundary layers, described the role of the relatively large spanwise variations brought about by streamwise vortices in the transition region; contrary to earlier work of Klebanoff *et al.* (1962), who found that peaks in r.m.s. longitudinal velocity occurred whenever the streamwise vortices involved motion away from the wall, in association with *horizontal* shear layers; Professor Tani described experiments of H. Komoda in which breakdown is preceded by a *vertical* shear layer at a position nearer to a valley. In some experiments conducted with small spanwise variations no velocity spikes (regions of high-frequency velocity fluctuation) appeared.

With regard to non-linear effects in the so-called parallel flows, J. T. Stuart paid particular attention to the 'subcritical effect' in plane Poiseuille flow; below the critical Reynolds number of linearized theory, oscillations can grow provided their amplitude lies above some threshold value. Calculations of Reynolds & Potter (1967) and of Pekeris & Shkoller (1967), based on the theory of Stuart (1960) and Watson (1960), have established this both for the case of constant pressure gradient and for that of constant mass flux. Moreover, Reynolds & Potter did a trial calculation suggesting that the generation of harmonics does not play an important qualitative role in this process; rather the important effects are the modification of the mean flow by the Reynolds stress of the oscillation, coupled with a consequent modification of the fundamental shear-wave oscillation itself. This confirms the assumption made earlier by Meksyn & Stuart (1951), and thus the important physical truth of this finite-amplitude process is established. Different speakers, especially W. C. Reynolds, emphasized the possible important influences of three-dimensionality and of any change with amplitude of the wavelength of the oscillation. These effects require further investigation.

W. C. Reynolds described the use of stability theory to suggest the importance of organized wave motions in turbulent shear flows of various kinds. A linear model was described, but the effect of background turbulence on the wave motions was kept through use of an eddy viscosity. Although the model for eddy viscosity was argued from rather basic mechanical principles, the upshot is that the governing wave stability equation is the Orr–Sommerfeld equation with a viscosity (here regarded as the *eddy* viscosity) which is a function of the coordinate in the direction of shear. Since the eddy viscosity is much higher than the molecular kinematic viscosity, the corresponding Reynolds numbers (eddy Reynolds numbers) based on eddy viscosity are much lower conventional Reynolds numbers.

Reynolds applied his theory to the two-dimensional flow behind a grid of equally spaced long circular cylinders, assuming a constant eddy viscosity, and found that the critical eddy Reynolds number (for neutral stability) was within a few per cent of the actual eddy Reynolds number inferred from experiments of Gran Olsson (1936). This led Reynolds to suggest the principle that the critical eddy Reynolds number in his theory gives the eddy viscosity of the actual turbulent flow. However, it can be argued, as Reynolds does in other cases, that the non-linear *growth* of the waves is also relevant (possibly to a quasi-equilibrium).

Reynolds's calculations indicated the large degree to which Reynolds stresses (and therefore energy transfer) can be affected by the eddy viscosity distribution. He applied the theory to the problem of wind-generated water waves, the calculated air Reynolds stress (according to the above theory) being too high by a factor of 10 compared with quoted experiments of R. H. Stewart. Direct nonlinear wave effects may be relevant. This line of investigation promises to be one of great importance, even though more work is still needed.

D. Coles described briefly his well-known experimental work (1965) on the development of complex wave motions between concentric circular cylinders when the inner is rotating and the outer is at rest. He emphasized the numerous hysteresis loops between different states obtaining by variations in rotation speeds. A primary problem here is that of explaining the instability of the Taylor vortices; J. T. Stuart described the work of Davey, DiPrima & Stuart (1968), which shows that the wavy vortex motion can arise as a natural instability of (or mathematical branching from) the Taylor vortex flow. Reasonable agreement with experiment was obtained. Many problems remain, both of a mathematical and a physical character, and this relatively simple flow configuration still presents a formidable test for non-linear theories of fluid processes.

Much interest in the non-linearity of fluid flow behaviour is directed, however, towards convection in its various forms, because of its importance in geophysics and chemical engineering. Several lectures were devoted to these topics, and in this section we are concerned with those relating to normal modes; effects of randomness will be described later.

E. L. Koschmieder showed photographs of convection patterns in a horizontal layer of fluid heated from below. 'Rolls' were produced. They were concentric circles when a container of circular shape was used, but with a square container square cells could appear. He emphasized that, when hexagonal cells occur, the driving mechanism is almost always the variation of surface tension with temperature, rather than the buoyancy force associated with an unstable temperature gradient in the vertical: a notable exception exists, however, in the experiments of Dr Ruby Khrishnamarti, who obtained hexagonal cells between parallel plates when the mean temperature was allowed to rise uniformly, and then buoyancy certainly produced hexagonality. Among other points emphasized by Dr Koschmieder is the need to evaluate the change to greater wavelengths with increase of amplitude, a phenomenon which may be associated with the presence of lateral boundaries. (A related problem defined by Coles is that of explaining the increase in Taylor vortex wavelength with amplitude in his experiments on circular Couette flow; here the finite length of the cylinders presumably corresponds to the lateral boundaries in convection.)

In further explanation of her experiments, Dr Khrisnamarti said that they were done with two conducting planes as upper and lower boundaries, the uniform rate of increase of the mean temperature being equivalent mathematically to a uniform heat source in the fluid. Hexagonal cells were produced by a subcritical or threshold, phenomenon, below the critical Rayleigh number. After a long time, however, the convection pattern in the box became one of parallel roll cells.

L. A. Segel gave an enlightening account of various situations in which nature gives rise to patterns of a hexagonal form, before going on to elucidate and describe in detail the evolution of hexagonal cell patterns in thermal convection. The presence of hexagons depends very much on vertical asymmetries due, for example, to the variation of surface tension or viscosity with temperature, and Dr Segel explained how, because of such variations, two-dimensional and appropriate 'rectangular' modes can reinforce themselves to produce hexagons even below the 'critical' Rayleigh number of linearized theory. As the Rayleigh number is raised hexagonal cells, which are preferred just below and above the 'critical' Rayleigh number, are replaced by two-dimensional roll cells. Further experimental checks of this theoretical result are required. One important way in which theory accounts for experiment is in their agreement that fluid movement on the cell's central axis is in the direction of increasing viscosity.

Dr Segel also indicated that recent work done by W. Eckhaus, R. C. DiPrima and himself shows that, if two modes of the same type (two-dimensional rolls) are allowed to interact, the non-linearity in the system spreads energy very strongly to many other wavelengths: in fact, it is not possible to restrict attention to, say, two modes only of different wavelengths.

In discussion of Segel's paper several speakers emphasized the role of the initial pre-convective condition, especially in relation to any discussion of a continuous spectrum of disturbance modes. Professor E. Palm in a later lecture also stressed the role of non-linear interactions in support of Segel's view.

D. Lortz described some theoretical work on the effects on convection of rotation about a vertical axis, where, once more, a subcritical phenomenon is possible, with a threshold amplitude for instability. Experimental evidence was illustrated at length by Dr T. Rossby. He found the experimental value of the rotation parameter (the Taylor number) beyond which a subcritical phenomenon occurs, to be much higher than the value given by Lortz's theory. There is some agreement, however, with Veronis's theory for 'free' (as contrasted with the experimental 'rigid') boundaries, as to the critical Taylor number defined above.

Convection between parallel planes at high Rayleigh number, when the cells are conceived as having an inviscid core surrounded by a boundary layer, was analyzed from a theoretical point of view by D. J. Tritton, especially, and by J. L. Robinson. Tritton's ideas were developed with particular reference to an attempt to show why convection cells have a greater lateral dimension when generated within a fluid with heat sources, than in the more conventional Bénard case of a fluid heated from below. Some success in this direction was reported. J. L. Robinson emphasized differences between his own model and those of Pillow (1949) and Tritton. The problem is still clouded by controversy, not least in reference to possible occurrence of a form of boundary layer separation. D. Coles drew attention to some recent work, due to P. Wesseling, extending and confirming Batchelor's (1960) analysis of the corresponding rotating cylinder problem (Taylor vortices at large Taylor number).

4. Turbulent–non-turbulent interfaces, transition and turbulence

It is well known that turbulent and transitional flows often of necessity possess regions of sharp change between laminar and turbulent zones of flow, or between non-turbulent (e.g. potential) and turbulent regions of flow. In his lecture Dr L. S. G. Kovasznay chose to emphasize the latter type of interface. First of all, he described the use of a new, conditional sampling technique for turbulence measurements, involving the use of two hot-wire probes. By assessing continually the state of flow (as to whether it was turbulent or not) he was able to derive from measurements the statistics of the flow on both sides of the turbulent–non-turbulent interface. This is especially relevant since, at the probe, the flow varies from turbulent to non-turbulent at different stages, as the contorted interface, or 'superlayer', pursues its random oscillations. The new technique enabled a model to be built up of the flow in the neighbourhood of the interface.

Among results found experimentally by Kovasznay are the following. Inside the interface the flow has nearly uniform mean vorticity, so that the velocity gradient there is uniform. There is no velocity jump so that the interface is not a layer of high vorticity or vortex sheet. It is in fact a layer on which there is sharp vorticity gradient, the exterior flow being irrotational. At a fixed point the mean velocity when there is turbulence is less than that when there is no turbulence by about 6 or 7 % of the speed of the free stream. Moreover, the magnitude of the turbulent velocity fluctuations within the boundary layer is of order twice that of the non-turbulent fluctuations outside.

Finally, Kovasznay suggested that the sublayer (on the solid wall) affected the interface (or superlayer) in the following way. Bursting events within the three-dimensional sublayer (of Kline *et al.* 1967) cause a lifting up of retarded fluid into the main body of the boundary layer. The superlayer is contorted by the necessity to ride over the 'lumps' of slightly retarded fluid.

Possible sequences of events by which a laminar flow can evolve to turbulence were treated by I. Tani, with reference to the work of P. S. Klebanoff, L. S. G. Kovasznay and their co-workers, as well as to his own experiments. This matter has been surveyed elsewhere (Stuart 1965) and here we refer to new knowledge arising since that date. As in the earlier work, streamwise vortices play a dominant role in the flow development but, in a contrasting feature revealed by Komoda (1967), transition may be preceded by strong concentrations of vorticity normal to the plate. This vertical shear layer appeared subsequently to the horizontal shear layer observed by the earlier experiments. Its importance for embryo turbulent spot development remains to be assessed. Professor Tani also reassured his audience that Tokyo turbulent spots are very similar to Washington turbulent spots, even in detail; for this assessment he used the conditional sampling technique of Kovasznay, which was mentioned earlier in this section. In another lecture on transition, Dr F. X. Wortmann discussed especially the application of the Tellurium technique for visualization of the 'peak valley' streamwise vortex system of Klebanoff, Kovasznay and Tani.

G. Lespinard outlined his experiments on the effects of boundary-layer suction on the transition processes described above. Broadly speaking suction does not affect the qualitative nature of the transition process, although it reduces the rate of growth of waves and of the associated longitudinal vortex system, and renders the actual occurrence of embryo turbulent spots less explosive. Clearly it would be desirable to relate the quantity of suction required to the known magnitudes of secondary flow (longitudinal vortex) velocities.

5. Thermal turbulence and turbulent shear flows

In a general lecture Dr A. A. Townsend described his outlook on turbulence at the present time, noting initially the importance of similarity by reference to P. Bradshaw's calculations of turbulent boundary layers by use of the energy equation. Dr Townsend described in detail the varied aspects of turbulence in boundary layers contrasted with the free turbulence in jets and wakes. The role of different eddy structures was noticed.

Dr John Elder gave an account of observations and computational experiments related to thermal turbulence in a horizontal layer of fluid heated from below. The seat of the genesis of the turbulence lay in small random motions within a sublayer on the lower boundary, in accordance with the ideas of Howard (1964). Randomness was deliberately introduced by Elder into his computer calculations. The computation illustrated the eruption and breakoff of 'turbulent' blobs from the lower boundary. Even though the computations were for two-dimensional flow there was considerable qualitative comparison with experimental observations.

Three speakers, L. N. Howard, F. Busse and W. V. R. Malkus, devoted their efforts to the remarkably effective line of enquiry, initiated by Howard (1963) some years ago, in which upper bounds are sought on some property, such as heat or momentum, which is transported by turbulence. Originally it was applied by Howard to thermal turbulence between two horizontal planes, in an attempt to justify rigorously the physico-mathematical ideas of W. V. R. Malkus on turbulent processes and structure. Although the original intention was not achieved, this type of analysis proved its own intrinsic worth. More recently, F. Busse, in particular, has applied related ideas to shear flow turbulence.

In a lecture summarizing the present state of these theories Howard explained how his ideas of 1963 have been modified. Then the upper bound or Nusselt number, obtained from an integral constraint subject to the continuity was found to be proportional to the $\frac{3}{8}$ power of the Rayleigh number. Stimulated by F. Busse, Howard included more than one wave-number in his calculations and obtained a power of $\frac{15}{23}$ or $\frac{53}{128}$ for 2 or 3 wave-numbers respectively! Both Howard and Malkus elaborated on other aspects of this work, while Busse applied developed ideas to shear flows as well as to thermal turbulence.

6. Other problems related to that of transition to turbulence

An interesting and stimulating lecture on free-surface effects was given by a chemical engineer, Dr L. E. Scriven, who described many aspects of wave motion, instability and turbulence, especially in relation to interfacial phenomena between different fluid phases. He drew attention to the importance of surface tension gradients, of the Gibbs surface elasticity, of compositional surface viscosity (associated with a soluble surfactant) and of structural surface viscosity (associated with monomolecular layers). He also drew attention to the little-studied problem of the formation of waves on rivulets, as on the windscreen of a car, for example. The complexity, over and above that of liquid films running on an inclined plane under gravity, is associated with the limited lateral expect of the rivulet. The now well-known phenomenon of surface-tension gradients as a mechanism for instability was also elaborated.

In discussion after this lecture, T. Brooke Benjamin quoted some observations on rivulets under a plane, namely that they could be unstable at low Reynolds numbers, stable at intermediate Reynolds numbers and turbulent at higher Reynolds numbers. In response to a question by M. J. Lighthill, Dr Scriven explained that 'active' stresses could be important in producing motion in small organisms.

In a remarkable film strip Dr Scriven showed a vortex ring incident on a free surface of water from below and being reflected as another vortex ring, whose direction of motion was inclined at a different angle from the first! He described this as 'the scattering of vortons by an interface by production of riplons'.

Dr P. G. Saffman explained his (1962) theory of instability in the flow of a dusty gas, and extended the ideas to a discussion of shear flow turbulence in a pipe. Interestingly he found a small increase in the mass flow due to the presence of the dust, with an increase in velocity gradient at the wall. This is in accordance with experimental evidence that the pressure drop in a pipe is reduced by dust, for a given mass flow.

Vortex breakdowns, and the multitudinous theories of their character, were described by Dr H. Ludwieg. He paid attention especially to his own theory of the breakdown, as being due to an instability of the swirling and translating flow, but referred also to the work of M. G. Hall (as a phenomenon analogous to 'separation'), H. B. Squire, Benjamin (conjugate flow theory) and Lambourne. This topic still stands unresolved, but is becoming increasingly dominated by Benjamin's conjugate-flow theory.

The phenomenon of water-wave breaking was catalogued and illustrated very ably by T. Brooke Benjamin, who drew attention to 'plunging' and 'spilling' forms of breaking, the former being the violent form with a vertical face, and the latter being a development out of the Stokes 120° angled wave. The role of Taylor instability at an accelerating interface was mentioned as a mechanism for bringing about 'turbulence' in wave breaking.

'Billows', at the interface of a two phase fluid system were illustrated by S. A. Thorpe, while J. W. Miles talked on the topic of instabilities of lee waves. Rigorous theorems on instability were considered by M. Cotsaftis, with characteristic verve.

A final summary lecture was given by M. J. Lighthill, drawing attention to many of the points posed above in a manner aimed at showing up connexions and interrelations between them as far as possible. This summary was later refined and extended to become a part of his general lecture on 'Turbulence' given at the Osborne Reynolds Centenary (Lighthill 1969).

Lectures presented at the NATO Advanced Study Institute

Benjamin, T. B. Wave breaking.

Busse, F. H. Upper bounds on the transport of heat, mass and momentum by turbulent flows.

Coles, D. Curved flows.

Cotsaftis, M. General theorems on stability.

Davis, S. H. On the principle of exchange of stabilities.

Elder, J. Convection in thermal turbulence.

Faller, A. Coriolis-driven instability.

Gaster, M. The development of three-dimensional wave packets in shear flows.

Howard, L. N. Thermal turbulence.

Koschmieder, E. L. Cellular convection.

Kovasznay, L. S. G. Turbulent and non-turbulent interfaces.

Lespinard, G. Effects of suction on the mechanism of boundary-layer transitions.

Lighthill, M. J. Concluding summary.

Lin, C. C. General survey.

Ludwieg, H. Vortex breakdown.

Malkus, W. V. R. Bounds on absolute stability.

Michalke, A. Instability in mixing regions.

Miles, J. W. Instabilities of lee waves.

Palm, E. Non-linear interactions.

Reynolds, W. C. The stability of organized waves in turbulent shear flows.

Rosenblat, S. Stability of time-dependent flows.
Rossby, T. Convection with rotation.
Saffman, P. G. Dusty gases.
Sato, H. Transition in wakes, jets and mixing regions.
Scriven, L. E. Free surface effects.
Segel, L. A. Emergence of cellular patterns.
Stuart, J. T. Non-linear effects in parallel flows.
Tani, I. Boundary layer transition.
Thorpe, S. Billows.
Townsend, A. A. Shear turbulence.
Tritton, D. J. Convection.
Wortmann, F. X. Visualization of transition.

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